A Fundamental Structure of Continuum Mechanics I. Elastic Systems

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Continuum mechanics and thermodynamics are complementary areas of classical field theory. Their fundamental structure, however, is rather different according to the present view: on the one hand there is the equation of motion or the equivalent balance of momentum, on the other there is Gibbs' fundamental equation.

In a preceding paper it has been shown that systems having a potential of acceleration possess a structure of continuum mechanics which corresponds to the one of thermodynamics. This discussion is extended to elastic systems. In this case a tensorial potential of acceleration must be introduced. The equation of motion derived for this extended structure contains two additional terms (reactive forces) not present in the conventional balance of forces.

Introduction

In the preceeding paper [1], systems possessing an acceleration potential (preferable is the name potential of inertia which will be used in the following) were shown to have a theoretical structure which corresponds to Gibbs' fundamental equation of thermodynamics and the relations derivable therefrom. The counterpart to the internal energy U is a quantity named proper mechanical energy (PME), sign A. Its density a is given by

$$a = (\phi + \varphi)\, m + \left(1 + \frac{\phi + \varphi}{c^2}\right) P \tag{1}$$

with

$$\mathrm{d}a = (\phi + \varphi)\, \mathrm{d}m. \tag{2}$$

From (1) and (2) results

$$\left(m + \frac{P}{c^2}\right) \mathrm{d}(\phi + \varphi) + \left(1 + \frac{\phi + \varphi}{c^2}\right) \mathrm{d}P = 0 \tag{3}$$

which, written with gradients, is the equation of motion[1]. ϕ is the potential of inertia (i.e. grad $\phi = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$; $\boldsymbol{v}$ velocity of a material point, t time), φ the gravitational potential (which may be time dependent), m the mass density, P the pressure, and c the velocity of light.

The thermodynamic equations (of a one-component system[2]) corresponding to (1), (2) and (3) are

$$u = T s - P + n \mu \tag{4}$$

with

$$\mathrm{d}u = T\, \mathrm{d}s + \mu\, \mathrm{d}n, \tag{5}$$

and the Gibbs-Duhe equation resulting from (4) and (5)

$$s\, \mathrm{d}T - \mathrm{d}P + n\, \mathrm{d}\mu = 0, \tag{6}$$

where u, the density of internal energy, is considered to contain the inert energy present in matter, i.e.

$$u = m c^2; \tag{7}$$

m then is the mass density which in relativistic dynamics is measured by an observer moving with the respective material point, T the absolute temperature, s the entropy density, n the density of mol number and μ the chemical potential.

Addition of the corresponding equations leads to

$$e = \left(1 + \frac{\phi + \varphi}{c^2}\right)(u + P) = \left(1 + \frac{\phi + \varphi}{c^2}\right)(T s + \mu n) \tag{8}$$

[1] The terms with c^2, negligible for ordinary systems (i.e. $P \ll m c^2$), are of structural interest, however. Notwithstanding their occurance the text is non-relativistic. They stem from the use of (7), which can be thought of as obtained by a calorimetric experiment, the constant being an empirical one. – The relations containing c are to be considered as limiting cases of a corresponding relativistic context.

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[2] The number of independent intensive variables determining a local state of a one-component system in general is two, and reduces to one in case of reversibility (local adiabatic behaviour).

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$$\mathrm{d}e = \left(1+\frac{\phi+\varphi}{c^2}\right)\mathrm{d}u$$

$$= \left(1+\frac{\phi+\varphi}{c^2}\right)(T\,\mathrm{d}s+\mu\,\mathrm{d}n) \tag{9}$$

and

$$s\,\mathrm{d}\left\{\left(1+\frac{\phi+\varphi}{c^2}\right)T\right\}$$

$$+n\,\mathrm{d}\left\{\left(1+\frac{\phi+\varphi}{c^2}\right)\mu\right\}=0, \tag{10}$$

where $e = a + u$ is the density of a quantity named proper energy (PE), sign E.

The peculiar properties of PE are:

1. it is conserved under conditions of a time dependent gravitational potential,
2. it is invariant with regard to arbitrarily accelerated frames of reference,
3. it does not change for a material system undergoing any process with its surroundings (deformation, acceleration, heat transfer, diffusion).

In [1] it has already been pointed out that a generalization of this structure necessitates the introduction of a tensorial potential of inertia ϕ_{ik} with $\frac{\partial\phi_{ik}}{\partial x_k}=\frac{\mathrm{d}v_i}{\mathrm{d}t}$ (indices with summation convention). The carrying through of this program is the content of this paper.

As model serves a generalization of Gibbsian thermodynamics [2], where it has been shown that, as soon as the pressure of a system becomes tensorial, at least one chemical potential (for an elastic system all chemical potentials) has to become a tensor of the same character.

For an elastic one-component system, (4) to (6) generalize to

$$u\,\delta_{ik}=T\,s\,\delta_{ik}-P_{ik}+n\,\mu_{ik} \tag{11}$$

with

$$\mathrm{d}u = T\,\mathrm{d}s - n\,\mu_{mn}\frac{\partial X_L}{\partial x_n}\,\mathrm{d}\frac{\partial x_m}{\partial X_L} \tag{12}$$

and

$$s\,\mathrm{d}T\,\delta_{ik}-\mathrm{d}P_{ik}+n\,\mathrm{d}\mu_{ik}-\mathring{P}_{ik}\frac{\mathrm{d}J}{J}$$

$$+\mathring{P}_{mn}\overline{\frac{\partial X_L}{\partial x_n}\,\mathrm{d}\frac{\partial x_m}{\partial X_L}}\,\delta_{ik}=0 \tag{13}$$

((12), (20) and (22) of [2]).

δ_{ik}, μ_{ik}, P_{ik} are components of the unit tensor, the tensorial chemical potential, and the pressure tensor, respectively (a circle above a tensor component indicates tracelessness; without indication all tensorial quantities are considered to be symmetric).

$\boldsymbol{r}(t) = \{x_1, x_2, x_3\}$ is the position vector of a material point, $\boldsymbol{R}(t) = \{X_1, X_2, X_3\}$ the position vector of the reference state, and $J = \left|\frac{\partial x_i}{\partial X_k}\right|$ the Jacobian of the deformation $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{R})$.

A Fundamental Equation for the Mechanics of Elastic Systems

In [1] the existence of the new structure was shown, starting from Euler's equation of motion. This cannot be done in the case of Cauchy's equation of motion of an elastic body. The scalar potential field $\phi(\boldsymbol{r}, t)$ (if it exists) can be derived from the vector field $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$, but not the tensor field $\phi_{ik}(\boldsymbol{r}, t)$ looked for.

The procedure therefore has to be a direct generalization of (1) to (3) with the equation of motion as a consequence.

The idea to be used runs parallel to the tensorial generalisation of the chemical potentials in thermodynamics already cited. There the non-spherically symmetric deformation of an elastic body leading to a tensorial pressure P_{ik} is connected with a non-spherically symmetric shift of matter (measured by molnumber density) inducing tensorial chemical potentials. Here, correspondingly, the shift of mass is considered, inducing a tensorial potential of inertia.

In contradistinction to [1] relation (7) is used from the start [3]. For elastic bodies its use is of no practical interest. It is done to expose the general structure. Neglect of the terms with c directly leads to the less general procedure.

In the case of scalar ϕ, from the equation of balance for internal energy (reversible deformation)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -u\frac{\partial v_i}{\partial x_i} - P\frac{\partial v_i}{\partial x_i} \tag{14}$$

with (7) follows

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -m\frac{\partial v_i}{\partial x_i} - \frac{P}{c^2}\frac{\partial v_i}{\partial x_i}. \tag{15}$$

[3] In [1], in a first step the existence of the approximate form of (1) and (3) (not containing the terms with c) is shown, to be generalized later on.

When (7) is used the distinction between m and n as variable quantities becomes essential, because then in $m = \hat{m}\,n$ the molar mass is variable too: $\hat{m}(s, n)$.

Introduction of (15) in (2) (formulated with material time derivatives) and addition of $-P\frac{\partial v_i}{\partial x_i}+P\frac{\partial v_i}{\partial x_i}$ on the right hand side leads to

$$\frac{da}{dt}=-\left\{(\phi+\varphi)\,m+\left(1+\frac{\phi+\varphi}{c^2}\right)P\right\}\frac{\partial v_i}{\partial x_i}+P\frac{\partial v_i}{\partial x_i}=-a\frac{\partial v_i}{\partial x_i}+P\frac{\partial v_i}{\partial x_i}, \tag{16}$$

the balance equation of PME.

In the elastic case, the balance equation for internal energy is

$$\frac{du}{dt}=-u\frac{\partial v_i}{\partial x_i}-P_{ik}\frac{\partial v_i}{\partial x_k} \tag{17}$$

(as P_{ik} is considered to be symmetric, so is $\frac{\partial v_i}{\partial x_k}$; here and in the following it stands for $\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)$).

With (7) follows

$$\frac{dm}{dt}=-m\frac{\partial v_i}{\partial x_i}-\frac{P_{ik}}{c^2}\frac{\partial v_i}{\partial x_k} \tag{18}$$

for the balance of mass.

The idea of non-spherical shift of mass then leads to

$$\frac{dm_{ik}}{dt}=-m\frac{\partial v_i}{\partial x_k}-\frac{P_{mn}}{3c^2}\frac{\partial v_m}{\partial x_n}\delta_{ik}. \tag{19}$$

The first term on the right hand side of (19) characterizes the non-spherically symmetric shift of mass. The source term of (18) remains scalar. Equation (18) is the trace part of (19).

If now this shift induces a tensorial potential of inertia, then in generalization of (2) there is

$$\begin{aligned}\frac{da}{dt}&=(\phi_{ik}+\varphi\,\delta_{ik})\frac{dm_{ik}}{dt}\\&=-(\phi_{ik}+\varphi\,\delta_{ik})\left(m\frac{\partial v_i}{\partial x_k}+\frac{P_{mn}}{3c^2}\frac{\partial v_m}{\partial x_n}\delta_{ik}\right)\\&=-(\phi_{ik}+\varphi\,\delta_{ik})\,m\frac{\partial v_i}{\partial x_k}-\frac{\bar\phi+\varphi}{c^2}P_{mn}\frac{\partial v_m}{\partial x_n}\\&=-\left\{(\phi_{ik}+\varphi\,\delta_{ik})\,m+\frac{\bar\phi+\varphi}{c^2}P_{ik}\right\}\frac{\partial v_i}{\partial x_n},\end{aligned} \tag{20}$$

where the conversions on the different stages are obvious. $\bar\phi=\frac{1}{3}\phi_{ii}$ is the scalar part of ϕ_{ik}.

Addition of $-P_{ik}\frac{\partial v_i}{\partial x_k}+P_{ik}\frac{\partial v_i}{\partial x_k}$ on the right hand side gives

$$\frac{da}{dt}=-\left\{(\phi_{ik}+\varphi\,\delta_{ik})\,m+\left(1+\frac{\bar\phi+\varphi}{c^2}\right)P_{ik}\right\}\cdot\frac{\partial v_i}{\partial x_k}+P_{ik}\frac{\partial v_i}{\partial x_k}. \tag{21}$$

As the conservation property of PE has to be fulfilled by (17) and (21),

$$\frac{de}{dt}=\frac{da}{dt}+\frac{du}{dt}=-e\frac{\partial v_i}{\partial x_i}, \tag{22}$$

(21) is the generalized balance equation for PME with

$$a\,\delta_{ik}=(\phi_{ik}+\varphi\,\delta_{ik})\,m+\left(1+\frac{\bar\phi+\varphi}{c^2}\right)P_{ik} \tag{23}$$

instead of (1). Equation (23) corresponds to (11). – (One may notice that (19) is the only reasonable generalization of (15) allowing (22).)

Equation (23) indicates that with P_{ik} also ϕ_{ik} has to be symmetric. Equation (23) can be split in

$$a=(\bar\phi+\varphi)\,m+\left(1+\frac{\bar\phi+\varphi}{c^2}\right)\bar P \tag{24}$$

with $\bar P=\frac{1}{3}P_{ii}$, and

$$0=\mathring{\phi}_{ik}\,m+\left(1+\frac{\bar\phi+\varphi}{c^2}\right)\mathring{P}_{ik}. \tag{25}$$

Besides the convection term there is no additional flux term in (21), in contradistinction to the balance equation of conventional mechanical energy (sum of kinetic and potential energy) with its $P_{ik}\,v_k$. This means that a travelling elastic wave neither transports PME nor PE. For an elastic collision follows: there is no exchange of PME between colliding bodies during a collision (only a shift from PME to internal energy and the reverse within each body), whereas PE remains unchanged for each body.

The term $\frac{\partial v_i}{\partial x_k}$ can be transformed according to

$$\frac{\partial v_i}{\partial x_k}=\left(\frac{\partial\left(\frac{\partial x_i}{\partial t}\right)_{\boldsymbol{R}}}{\partial x_k}\right)_t=\left(\frac{\partial\left(\frac{\partial x_i}{\partial t}\right)_{\boldsymbol{R}}}{\partial X_L}\right)_t\left(\frac{\partial X_L}{\partial x_k}\right)_t=\left(\frac{\partial X_L}{\partial x_k}\right)_t\left(\frac{\partial\left(\frac{\partial x_i}{\partial X_L}\right)_t}{\partial t}\right)_{\boldsymbol{R}}=\frac{\partial X_L}{\partial x_k}\frac{d\left(\frac{\partial x_i}{\partial X_L}\right)_t}{dt}. \tag{26}$$

Hence (20) (in the last form) can be written

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\left\{(\phi_{ik} + \varphi\,\delta_{ik})\,m + \frac{\bar{\phi}+\varphi}{c^2}\,P_{ik}\right\} \cdot \frac{\partial X_L}{\partial x_k}\,\frac{\mathrm{d}\left(\dfrac{\partial x_i}{\partial X_L}\right)}{\mathrm{d}t}. \tag{27}$$

As a as well as $\dfrac{\partial x_i}{\partial X_L}$ are functions of $\boldsymbol{r}$ and t, there also exist the corresponding equations with local time derivatives and gradients, respectively. Therefore, generally there is

$$\mathrm{d}a = -\left\{(\phi_{ik} + \varphi\,\delta_{ik})\,m + \frac{\bar{\phi}+\varphi}{c^2}\,P_{ik}\right\} \cdot \frac{\partial X_L}{\partial x_k}\,\mathrm{d}\frac{\partial x_i}{\partial X_L}. \tag{28}$$

By use of (25) and

$$\mathrm{d}m = -\left\{\left(m + \frac{\bar{P}}{c^2}\right)\delta_{ik} + \frac{\mathring{P}_{ik}}{c^2}\right\}\frac{\partial X_L}{\partial x_k}\,\mathrm{d}\left(\frac{\partial x_i}{\partial X_L}\right), \tag{29}$$

following from (18) and (26), (28) can be transformed to

$$\mathrm{d}a = (\bar{\phi}+\varphi)\,\mathrm{d}m + \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right) \cdot \mathring{P}_{ik}\,\overline{\frac{\partial X_L}{\partial x_k}\,\mathrm{d}\frac{\partial x_i}{\partial X_L}}^{\circ}. \tag{30}$$

Equation (30) generalizes (2) and corresponds to (12), which by use of the traceless part of (11) can be written

$$\mathrm{d}u = T\,\mathrm{d}s + \bar{\mu}\,\mathrm{d}n - \mathring{P}_{ik}\,\overline{\frac{\partial X_L}{\partial x_k}\,\mathrm{d}\frac{\partial x_i}{\partial X_L}}^{\circ} \quad \text{with } \bar{\mu} = \frac{1}{3}\,\mu_{ii}. \tag{31}$$

The density of PE is given with (24) by

$$e = a + u = \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)(u+P). \tag{32}$$

Its differential results from (30) with (7) and (31) in

$$\mathrm{d}e = \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\left(\mathrm{d}u + \mathring{P}_{ik}\,\overline{\frac{\partial X_L}{\partial x_k}\,\mathrm{d}\frac{\partial x_i}{\partial X_L}}^{\circ}\right) = \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)(T\,\mathrm{d}s + \bar{\mu}\,\mathrm{d}n). \tag{33}$$

Equation (30) together with the differential of (24)

$$\mathrm{d}a = (\bar{\phi}+\varphi)\,\mathrm{d}m + m\,\mathrm{d}(\bar{\phi}+\varphi) + \mathrm{d}\left\{\left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\bar{P}\right\}$$

leads to

$$m\,\mathrm{d}(\bar{\phi}+\varphi) + \mathrm{d}\left\{\left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\bar{P}\right\} - \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\mathring{P}_{ik}\,\overline{\frac{\partial X_L}{\partial x_k}\,\mathrm{d}\frac{\partial x_i}{\partial X_L}}^{\circ} = 0, \tag{34}$$

the equation for $\bar{\phi}$ (whereas $\mathring{\phi}_{ik}$ is given by (25)).

Addition of (34) and the differential of (25)

$$m\,\mathrm{d}\mathring{\phi}_{ik} + \mathring{\phi}_{ik}\,\mathrm{d}\,m + \mathrm{d}\left\{\left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\mathring{P}_{ik}\right\} = 0$$

gives

$$m\,\mathrm{d}(\phi_{ik} + \varphi\,\delta_{ik}) + \mathrm{d}\left\{\left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)P_{ik}\right\} - \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\mathring{P}_{mn}\,\overline{\frac{\partial X_L}{\partial x_n}\,\mathrm{d}\frac{\partial x_m}{\partial X_L}}^{\circ}\,\delta_{ik} + \mathring{\phi}_{ik}\,\mathrm{d}\,m = 0. \tag{35}$$

The last member of (35) follows from (25) and (29) according to

$$\mathring{\phi}_{ik}\,\mathrm{d}\,m = \left(1+\frac{\bar{\phi}+\varphi}{c^2}\right)\mathring{P}_{ik}\left(\delta_{mn} + \frac{P_{mn}}{m\,c^2}\right)\frac{\partial X_L}{\partial x_n}\,\mathrm{d}\,\frac{\partial x_m}{\partial X_L}. \tag{36}$$

Equation (35) generalizes (3) and corresponds to (13).

Within the structure developed, the equation of motion of an elastic system results from (35) together with (36) (written with gradients) by tensorial contraction according to

$$m\frac{\mathrm{d}v_i}{\mathrm{d}t} + m\frac{\partial\varphi}{\partial x_i} + \frac{\partial P_{ik}}{\partial x_k} - \mathring{P}_{mn}\,\frac{\partial X_L}{\partial x_n}\,\overline{\frac{\partial}{\partial X_L}\frac{\partial x_m}{\partial X_L}}^{\circ} + \frac{\mathring{P}_{ik}}{J}\,\frac{\partial J}{\partial x_k} = 0 \tag{37}$$

where the terms with c^2 are already neglected and

$$\frac{\partial\phi_{ik}}{\partial x_k} = \frac{\mathrm{d}v_i}{\mathrm{d}t} \text{ and } \frac{\partial X_L}{\partial x_n}\,\overline{\frac{\partial}{\partial X_L}\frac{\partial x_m}{\partial x_k}}^{\circ}\,\delta_{mn} = \frac{1}{J}\,\frac{\partial J}{\partial x_k} \text{ is used.}$$

The last two members of (37), originating from the internal conditions of a system (reactive forces), do not appear in the conventional balance of forces. Their origin is the traceless tensor of inertia. In the course of

calculation they stem from members having the factor $m\,\mathring{\phi}_{ik}$, which according to (25) is substituted by $\mathring{P}_{ik}$. They allow to test the framework presented in this paper. Their existence should be looked for in the domain of large deformations. Only there the gradients will cause a noticeable contribution.

Elasticity with Heat Conduction

In the preceeding section reversible (locally adiabatic) systems are discussed. With heat conduction irreversibility comes in. The procedure for elastic systems runs along the same lines as in section IX of [1].

In (63) of [1], the equation defining the velocity of a material point when the momentum of the heat flux vector is taken into account, P has to be replaced by $\bar{P}$:

$$\left(m+\frac{\bar{P}}{c^2}\right)v_i=\left(m+\frac{\bar{P}}{c^2}\right)v_i^*+\frac{q_i}{c^2}; \tag{38}$$

v_i^* and q_i are components of the velocity of matter alone and of the heat flux vector respectively.

The equations of balance of the various quantities, (64) to (70), remain unchanged. Equations (67), (68) and (70) introduced in the second equation of (33) of this paper (which only differs from the corresponding one of [1], (57), by $\bar{\phi}$ and $\bar{\mu}$ instead of ϕ and μ) leads to the source term π_S of entropy:

$$\pi_S=-\frac{q_i}{\left(1+\dfrac{\bar{\phi}+\varphi}{c^2}\right)T^2}\,\frac{\partial\left(1+\dfrac{\bar{\phi}+\varphi}{c^2}\right)T}{\partial x_i}. \tag{39}$$

It has the same form as (71) of [1] with only $\bar{\phi}$ instead of ϕ. Elimination of $1+\dfrac{\bar{\phi}+\varphi}{c^2}$ by (34) results in

$$\pi_S=-\frac{q_i}{T^2}\cdot\left\{\frac{\partial T}{\partial x_i}-\frac{T}{m\,c^2+\bar{P}}\left(\frac{\partial\bar{P}}{\partial x_i}-\mathring{P}_{mn}\,\frac{\partial}{\partial X_L}\,\overset{\circ}{\frac{\partial x_m}{\partial X_L}}\,\frac{\partial x_n}{\partial x_i}\right)\right\}. \tag{40}$$

In (71) to (77) of [1] only P, ϕ and μ, where they appear, have to be replaced by $\bar{P}$, $\bar{\phi}$ and $\bar{\mu}$, respectively. These equations relate the flux terms, caused by the momentum of heat flux (according to (38)) of the various quantities, to the heat flux vector. For the conservation equation of PE then follows the same form as in the reversible elastic case, (22). This means that a material volume does not change its PE with heat transfer, a property of the generalized PE to be expected after the corresponding result in [1].

Only the equation of balance of internal energy, (66) of [1],

$$\frac{\mathrm{d}u}{\mathrm{d}t}=-u\,\frac{\partial v_i}{\partial x_i}-\frac{\partial}{\partial x_i}\{u(v_i^*-v_i)+q_i\}+\pi_U \tag{41}$$

(π_U is the source term of internal energy) needs a separate discussion. It has to be shown that (41) can be transformed to (17), as (17) is one of the equations the generalization to elastic systems starts from.

To this purpose, (22) and (41) are introduced in the first equation of (33) (written with material time derivatives). Then by use of (32) follows

$$\pi_U=\frac{\partial}{\partial x_i}\{u(v_i^*-v_i)+q_i\}-P_{ik}\,\frac{\partial v_i}{\partial x_k}. \tag{42}$$

Equation (41) together with (42) leads to (17).

[1] B. Stuke, Z. Naturforsch. **48a**, 883 (1993).
[2] B. Stuke, Z. Naturforsch. **30a**, 1433 (1975).